MATH1019 Mathematics for Engineers 2

Additional resources created by

Mathematics Education Support Hub (MESH)

ORDINARY DIFFERENTIAL EQUATIONS

This supplementary resource was created as YouTube videos aligned with your content. By watching these videos and practising the sample questions, students can build the knowledge required for Mathematics for Engineers 2. Students are encouraged to revise basic integration techniques in Modules 6, 8, and 9 of MATH1016 Mathematics for Engineers 1 before practising the questions on this sheet.

Contents

Module 1 - First Order Differential Equations

This worksheet consists of the classification of Ordinary Differential Equations (ODEs), separable and linear ODEs, and applications of ODEs.

1.1 Classification of differential equations (ODEs)

The video below introduces differential equations and demonstrates the classifications.

https://youtu.be/6o7b9yyhH7k?si=xOJS_edEiLSmHN22

1.2 Separable ODEs

The video below demonstrates how to solve separable ODEs.

https://youtu.be/C7nuJcJriWM?si=In4VMpycbvovg3TG

The video also demonstrates how to solve the following separable ODEs.

$a)\ \frac{dy}{dx} = \frac{x^2}{y^2}$ $b)\ y' = xy, y(0) = 5$ $c)\ \frac{dy}{dx} = y^2 + 1, y(1) = 0$

1.3 Linear ODEs

The video below demonstrates how to solve linear ODEs using an integrating factor.

https://youtu.be/gd1FYn86P0c?si=eQc65UAJ909-0gxp

The video also demonstrates how to solve the following linear ODEs using an integrating factor.

$a)\ y' + 2y = 2e^x$ $b)\ xy' + 4y = 2x^3$

$c)\ (x-2)y' + y = x^2 - 4$ $d)\ \frac{dy}{dx} = \frac{x^2 - y}{x}$

1.4 Applications of first-order differential equations

The questions in this section demonstrate how to solve applications of ODEs in exponential growth, decay, Newton's Law of Cooling, and flow-in-and-out rate problems.

i) Exponential growth

https://youtu.be/HDP0_6lODgU?si=06AujDneVJsnjDUL

The population of a small town grows in proportion to its current population. The initial population was 5000, and it grows 4% per year. This can be modelled by;

$$\frac{dP}{dt} = 0.04P, P(0) = 5000$$

a) Find an equation to model the population
b) Determine the population after three years
c) Determine how long it will take the population to double

i) Exponential decay

https://youtu.be/b53nhCEXlGI?si=Fx4EAAG8TvPDL_kD

Polonium-218 decays exponentially. The decay rate is 23.1% per minute and can be modelled by

$$\frac{dP}{dt} = -0.231P$$

a) Find an equation that satisfies the differential equation. Let $P(0) = P_0$.
b) If 1000 grams of polonium–218 is present at $t = 0$, how much is present after 90 seconds?
c) What is the half-life of Polonium-218?

ii) Newton's Law of Cooling

https://youtu.be/wEUuaB1esN4?si=KpRWjzeCxDRy8tVu

A cup of coffee at 190° F is left in a room of 70° F. At time $t = 0$, the coffee is cooling at 15° F per minute.

a) Find the function that models the cooling of the coffee.
b) How long will it take for the temperature to reach 143° F

iii) Flow In/ Out Flow Rate problems

https://youtu.be/eJPpFRGqHQo?si=d2a7Xz8hahD34yAD

A large tank holds 200 gallons of brine solution with 40 lbs of salt. A concentration of 2 lb/gal is pumped in at 4 gal/min, and the concentration leaving the tank is pumped out at 4 gal/min.

a) How much salt is in the tank after 1 hour?
b) How much salt is in the tank after a very long time?

iv) Electrical Circuits

https://youtu.be/i2mwUVwyAyM?si=XaidoUWlo9ceEfv1

Two 9V batteries are connected in series, with an inductance of ¼ Henry and a resistance of 8 Ohms. If the initial current is zero, determine the current.

Module 2 - Second-order linear ODEs

2.1 Introduction to second-order linear differential equations

The video in the link below introduces second-order differential equations. It explains the characteristics of homogeneous and non-homogeneous second-order linear differential equations with some relevant examples.

https://youtu.be/oFuz9YEERYw?si=cViLRiOL98EFgbDE

2.2 Homogeneous second-order linear ODEs with constant coefficients

The video below demonstrates how to solve homogeneous second-order linear ODEs with constant coefficients.

https://youtu.be/uI2xt8nTOlQ?si=PW0FQTMP7oZfOjkI

It explains how to answer the following questions on homogeneous second-order linear ODEs with constant coefficients.

$a)\ y'' - 5y' + 6y = 0$

$b)\ y'' - 6y' + 9 = 0$

$c)\ 9\dfrac{d^2y}{dx^2} + 24\dfrac{dy}{dx} + 16 = 0$

$d)\ y'' + 4y = 0, y(0) = 4 \text{ and } y'(0) = 6$

$e)\ y'' - 2y' + y = 0, y(0) = 3 \text{ and } y(1) = 7e$

2.3 Non-homogeneous linear ODEs with constant coefficients

The video below demonstrates how to solve non-homogeneous second-order linear ODEs with constant coefficients.

https://youtu.be/P3fc6v191mA?si=98Bk-bMJGTkv66j2

It explains how to answer the following non-homogeneous second-order linear ODEs with constant coefficients questions.

$a)\ y'' + 5y' + 6y = x^2$

$b)\ y'' + 9y = e^{2x}$

$c)\ y'' + 3y' + 2y = \cos x$

$d)\ y'' + y = \cos x$

$e)\ y'' - 9y = xe^x + \sin(2x)$

2.4 Applications of Second-order Linear ODEs

2.4.1 Spring mechanical motion

Free Oscillator Problem 1 – Vertical motion

The video at the link below demonstrates the behaviour of a free vertical oscillator and how to apply linear, second-order homogeneous ODEs with constant coefficients to model the motion.

https://youtu.be/V9bl02Ffo_o?si=e0vmXq1SbOjbXM6u

Free Oscillator Problem 2 – Horizontal motion

A spring with a mass of 2 kg has a natural length of 0.5 m. A force of 25.6N is required to maintain it stretched to 0.7m. If the spring is stretched to 0.7m and then released with initial velocity 0, find the position of the mass at any time t.

https://youtu.be/jZRCfnew088?si=qd22s_oU0f4_BJqP

2.4.2 LCR circuits

Problem 1 (homogeneous second-order linear ODEs with constant coefficients)

https://youtu.be/kUsGuWTIRQU?si=_B7jg4skMiARKGxV

A condenser of capacity C discharged through an inductor L and resistance R in series, and the charge q at time t satisfies the equation,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

Given that $L = 0.25\ Henrics, R = 250\ Ohms, C = 2 \times 10^{-6}\ Farads$. When $t = 0$, the charge q is 0.002 Coulombs and the current $\frac{dq}{dt} = 0$. Obtain the value of q in terms of 't'.

Problem 2 (non-homogeneous second-order linear ODEs with constant coefficients)

https://youtu.be/GHWM1KEvtrY?si=MgBMe6B2PMwDG_64

the charge $q(t)$ on the capacitor is given by the differential equation,

$$10\frac{d^2q}{dt} + 120\frac{dq}{dt} + 1000q = 17\sin 2t$$

At time zero, the current is zero, and the charge on the capacitor is $\frac{1}{2000}$ Coulombs. Find the charge on the capacitor for $t > 0$.

Module 3 - More second-order linear differential equations

3.1 Euler-Cauchy equations

The video in the link below introduces the Cauchy-Euler Differential Equations and how to solve them depending on the nature of the auxiliary equation, with an example.

https://youtu.be/oqpc89Lfx-E?si=MFskKGyXOVoPv7nr

Question	Solution
$x^2y'' + 11xy' + 25y = 0$	https://youtu.be/rV2K66M0Hk0?si=OizPo56nK7IOkrJX
$x^2y'' - 3xy' - 9y = 0$	https://youtu.be/X2mX-jGs7bM?si=KeJjMWAl0F9ETxNv
$x^2y'' + xy' + 36y = 0$	https://youtu.be/Umn4pc-dv1o?si=vtja3i8uWmxURYaB
$49x^2y'' + 49xy' + y = 0$	https://youtu.be/1LHo2dQOIR4?si=WdEbgKokvmdwGfkM
$x^2y'' - 2y' = 0$	https://youtu.be/lK8gYM3g0r4?si=GSLjUKtyRXqjSlFW
$3x^2y'' + 6xy' + y = 0$	https://youtu.be/N2PJVsV8dac?si=e4gtJcOpFqeFrKJF

3.2 Variation of parameters

The video below explains the variation of parameters method.

https://youtu.be/B0R3DgUqb0A?si=ymC5BfrRod1Y_V2F

The following videos demonstrate how to find the solutions for each second-order differential equation listed in the table using the variation of parameters method.

Question	Solution
$y'' + 4y' + 3y = xe^x$	https://youtu.be/_v5wq2ZXPUw?si=aqwEucySjFtbHNRU
$y'' - 5y' + 4y = e^{3t}$	https://youtu.be/UfZVBWO30M4?si=AM_XjWNLE7Xs3865
$y'' + y = \cot x$	https://youtu.be/mAvU3ApyUG4?si=LELBwiiHJhuGEOvw
$y'' - 5y' + 4y = 4t$	https://youtu.be/40tAOvUtRkA?si=Vq4jG_JmSpG2wB0K
$y'' + y = 1 + \tan t$	https://youtu.be/21UukhcDiDw?si=RRkppll_wnldy3gQ

3.3 Power series

The video below explains the process and introduces the power series method for solving second-order differential equations.

https://youtu.be/-fyzU8NyCG4?si=Omwh8afQbvtVkSck

The following videos show how to use the power series method to find solutions to each second-order differential equation listed in the table.

Question	Solution
$y'' + y + xy = 0$	https://youtu.be/JqPObdCVQUw?si=1HA2opgkYta0n58y
$y'' - (1 + x)y' = 0$	https://youtu.be/NLcgsJUSv3Q?si=XUHtU2o5Em5xR3Xx
$y'' - x^2y = 0$, Given that $y(0) = 1$ and $y'(0) = \frac{1}{2}$	https://youtu.be/4_pR2Sog9CA?si=LWR0rR2r7bbZ8uwo

Module 4 - Higher order ODEs

4.1 Introduces the method of solving higher-order linear ODEs

The video linked below explains the characteristics of higher-order differential equations and solving techniques.

https://youtu.be/7vwDp94wEhg?si=UUL0fMaGAXb9_m8W

4.2 Higher-order ODEs: Distinct roots

The video in the link below demonstrates how to solve higher-order constant-coefficient ODEs. The video comprises the following two questions.

https://youtu.be/is0F0u62IbY?si=XechQRFXPTiesrWX

a) $y'''' + y' = 0$

b) $y'''' - 3y''' + 3y'' - y' = 0$

4.3 Higher-order Euler-Cauchy ODEs

The video in the links below demonstrates how to solve higher-order Euler-Cauchy ODEs for the problems listed below.

Question	Solution
$x^3 \dfrac{d^3y}{dx^3} + 5x^2 \dfrac{d^2y}{dx^2} + 7x \dfrac{dy}{dx} + 8y = 0$	https://youtu.be/Wh7PxYSiys4?si=gk59fQ8PnTlGCQxg
$x^4 y'''' + 6x^3 y''' + 15x^2 y'' + 9xy' + 16y = 0$	https://youtu.be/CZHH9W_zqZI?si=rxwyTBQWsv8QL_mo

4.4 Higher order non-homogeneous linear ODEs

4.4.1 Method of undetermined coefficients: the video in the link below demonstrates how to solve the above-listed higher-order non-homogeneous linear ODEs with undetermined coefficients.

https://youtu.be/vJL3OPzZRT4?si=OxPoUBiOjd7f6Iov

a) $y''' + y' + y = e^{-t} + 4t$
b) $y'''' - 4y'' = e^t + t^2$
c) $y'''' + 2y'' + y = 3 + \cos(2t)$
d) $y''' + 4y' = t, y(0) = 0, y'(0) = 0, y''(0) = 1$

4.4.2 Method of variation of parameters: The video in the link below demonstrates how to solve the following higher-order nonhomogeneous linear ODEs with the method of variation of parameters.

https://youtu.be/U6HFSX546O8?si=-q8OLV-mafZE7FsE

4.5 Systems of homogeneous linear first-order ODEs

The video below demonstrates how to solve a system of homogeneous linear ODEs with constant coefficients.

https://youtu.be/ASYKt0P7I9c?si=OnPyrghjAYnJ7bEn

Student activity 1:

Consider the system of homogeneous linear ODEs with constant coefficients given by

$$\dot{x_1} = ax_1 + bx_2$$

$$\dot{x_2} = cx_1 + dx_2$$

Prove that the eigenvalues of the resulting characteristic equation are real.

Model solution:

1. The odes in matrix form are given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and the characteristic equation of the matrix is given by

$$\lambda^2 - (a+b)\lambda + ab - c^2 = 0.$$

The roots of the characteristic equation are

$$\lambda_{\pm} = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4c^2}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4c^2}}{2}.$$

Since the discriminant is non-negative, both roots are real (and distinct provided $a \neq b$ or $c \neq 0$).

1. With $\mathrm{A} = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}$, the characteristic equation is $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$, with roots $\lambda_1 = 1$ and $\lambda_2 = -2$. The corresponding eigenvectors are $\mathrm{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathrm{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is $\mathrm{x} = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, or $x_1 = c_1 e^t + c_2 e^{-2t}$ and $x_2 = -c_1 e^t + 2c_2 e^{-2t}$.

4.6 Solving a system of homogeneous linear ODEs with constant coefficients

4.6.1 Distinct real eigenvalues: The video below demonstrates how to solve systems of homogeneous linear odes with constant coefficients and distinct real eigenvalues, which demonstrates how to solve the following problem.

Find the general solution of

$$\dot{x_1} = x_1 + x_2$$

$$\dot{x_2} = 4x_1 + x_2$$

https://youtu.be/8e_ux_E0dJ0?si=EMGHF-nHtygAYYQv

Student activity 2:

Using matrix algebra, find the general solution of

$$\dot{x_1} = -x_2$$

$$\dot{x_2} = -2x_1 - x_2$$

Model solution

1. With $A = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}$, the characteristic equation is $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$, with roots $\lambda_1 = 1$ and $\lambda_2 = -2$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is $x = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, or $x_1 = c_1 e^t + c_2 e^{-2t}$ and $x_2 = -c_1 e^t + 2c_2 e^{-2t}$.

4.6.2 Complex-conjugate eigenvalues: The video below demonstrates how to solve systems of homogeneous linear ODEs with constant coefficients with complex-conjugate eigenvalues.

https://youtu.be/NnK_mJYIrxk?si=RzzJ-b5WxHHb2PyX

The video above shows how to solve the following problem.

Find the general solution of

$$\dot{x_1} = -\frac{1}{2}x_1 + x_2$$

$$\dot{x_2} = -x_1 - \frac{1}{2}x_2$$

Student activity 3:

Using matrix algebra, find the general solution of

$$\dot{x_1} = x_1 - 2x_2$$

$$\dot{x_2} = x_1 + x_2$$

Model Solution

1. With $A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 2\lambda + 3 = 0$, with root $\lambda = 1 + i\sqrt{2}$ and its complex conjugate. The corresponding eigenvectors are $v = \begin{pmatrix} 2i \\ \sqrt{2} \end{pmatrix}$ and its complex conjugate. The general solution is constructed from the linearly independent solutions $X_1 = \mathrm{Re}\left\{ve^{\lambda t}\right\}$ and $X_2 = \mathrm{Im}\left\{ve^{\lambda t}\right\}$, and is

$$\mathrm{x} = e^t \left(A \begin{pmatrix} -2\sin(\sqrt{2}\,t) \\ \sqrt{2}\cos(\sqrt{2}\,t) \end{pmatrix} + B \begin{pmatrix} 2\cos(\sqrt{2}\,t) \\ \sqrt{2}\sin(\sqrt{2}\,t) \end{pmatrix} \right),$$

or

$$x_1 = e^t \left(-2A\sin(\sqrt{2}\,t) + 2B\cos(\sqrt{2}\,t) \right), \qquad x_2 = e^t \left(\sqrt{2}A\cos(\sqrt{2}\,t) + \sqrt{2}B\sin(\sqrt{2}\,t) \right).$$

4.7 Phase plane, critical points and stability analysis

The video below explains what phase planes are and how to sketch them once you have the solutions to the differential equation. It also demonstrates how to determine critical points using the following examples.

https://youtu.be/FjLfRlrjCzw?si=rzuV6IXb1fd8JymN

Example 1 $\dot{x} = 4x - 3y$

$\dot{y} = 6x - 7y$

Example 2 $\dot{x} = -2x + y$

$\dot{y} = -4x$

Besides demonstrating the solutions for the above two examples, the video also explains five types of critical points: improper node, proper node, saddle point, centre point, and spiral point, depending on the solution of the eigenvalues.

Stable point - both λ completely complex

Spiralling in – complex λ, but the real part is negative

Spiral out – complex. λ, but the real part is positive

Straight in – both λ real and negative

Straight out – both λ real and positive

Saddle point – both λ, but one positive and one negative

4.7.1 Phase portrait of a stable or unstable node

Stable node (stable, attractive points): The video below demonstrates an example of a stable node and how to visualise it after finding the solution.

https://youtu.be/PMwxI5cEY7Y?si=BDNxpuie-BW0-bNw

The example used in the video is given below.

$$\dot{x_1} = -3x_1 + \sqrt{2}x_2$$

$$\dot{x_2} = \sqrt{2}x_1 - 2x_2$$

Saddle points: The video below demonstrates an example of a saddle point and how to visualise it after finding the solution.

https://youtu.be/6kaNxxEOElM?si=PIKO66foGlWfrh58

The example used in the video is given below.

$$\dot{x_1} = x_1 + x_2$$

$$\dot{x_2} = 4x_1 + x_2$$

4.7.2 Applications of Systems of Homogeneous Linear First-order ODEs - Mixing problems involving two tanks

Example 1

Two tanks are connected by pipes. Each tank contains 500 gallons of a salt solution. Through one pipe, the solution is pumped from the first tank to the second at 1 gal/min. Through the other, the solution is pumped at the same rate from the second tank to the first. Suppose that at time t=0, there is no salt in the tank on the right and 100 lb in the tank on the left. Find the salt content in each tank as a function of time.

https://youtu.be/aVbnPvngM98?si=4_nokpYAPHVakNva

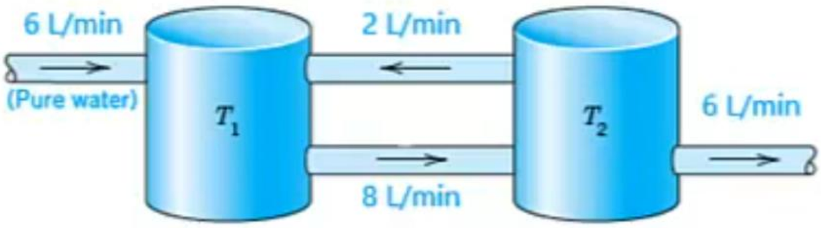
Example 2

Two tanks contain 360 litres of salt solution. Pure water pours into Tank A at a rate of 5 L/min. There are two pipes connecting Tank A to Tank B. The first pumps salt solution from Tank B into Tank A at a rate of 4 L/min. The second pumps salt solution from Tank A into Tank B at the rate of 9 L/min. Finally, there is a drain on Tank B from which salt solution drains at a rate of 5 L/min. Thus, it maintains a constant volume of 360 litres of salt solution each time. Initially, 60 kg of salt was present in Tank A, but Tank B contained pure water.

https://youtu.be/5ycX1cN-gS4?si=vXDoYdP8xEDIhVob

Example 3

Two large tanks, each holding 24 litres of a brine solution, are interconnected by pipes, as shown in the figure. Fresh water flows into the tank T_1 at a rate of 6 L/min, and fluid is drained out of the tank T_2 at the same rate; also, 8 L/min of fluid is pumped from the tank T_1 to tank T_2, and 2 L/min from tank T_2 to tank T_1. The liquids inside each tank are kept well stirred. If, initially, the brine solution in the tank T_1 contains 1 kg of salt and that in the tank T_2 contains 6 kg of salt, determine the mass of salt in each tank at time $t > 0$.



https://youtu.be/sVPRu8zzthg?si=ZCjE_tki2WL0_QwK

Module 05 - Laplace Transform

Note: The playlist link below demonstrates Laplace Transformation with different examples. It comprises 23 videos and totals under 230 minutes. This playlist covers all the content that you need for modules 6 and 7. Feel free to watch them, dive into this topic, and learn from the beginning. Otherwise, the videos below cover the contents of module 6, following the subtopics listed below from 6.1.

https://youtube.com/playlist?list=PLnVYEpTNGNtVH5YLVJsA2WxWXk6bAps-D&si=aYa1pGvSg4HxT-ek

5.1 - Introduction to Laplace Transform

The video in the link below demonstrates the introduction to the Laplace Transformation and shows how to apply it to $\mathcal{L}[e^{\lambda t}]$.

https://youtu.be/8oE1shAX96U?si=CNrR0zZKE1-TLi-I

The video in the link below shows a table of Laplace Transforms and their inverses.

https://youtu.be/5_lTZyU_BCw?si=TFZPTIrnfhNhD1VH

The links below show how to apply Laplace transforms to the problems in the table below.

Problem	Link to the model solution video
$\mathcal{L}[t^n], n > 0$	https://youtu.be/CERKluu27uI?si=sLgzy8HMB-_VxqI-
$\mathcal{L}[\cos(\beta t)]$	https://youtu.be/mwWjCNITvc8?si=cq704nYTwkFa4lTi

5.2 Basic Transforms

The videos linked in the table below demonstrate how to apply the Laplace Transform to the following questions.

Problems		Link to the video solution
a) b) c)	$\mathcal{L}[t^4]$ $\mathcal{L}[5e^{2t} - t^3]$ $\mathcal{L}[t^2 - 7 + \cos(2t)]$	https://youtu.be/CT4OMdCCnz4?si=nqOyY1QWdO0MIVrS
a) b) c)	$\mathcal{L}[3 - e^{-3t} + 5\sin(2t)]$ $\mathcal{L}[3 + 12t + 42t^3 - 3e^{2t}]$ $\mathcal{L}[t + 1)(t + 2)]$	https://youtu.be/0t9etH9CnRg?si=81gsQCmqgr2ft1Cx

5.3 First Shifting Theorem

The videos in the links below explain the First Shifting Theorem and demonstrate how to apply it to solve the following questions.

Problems		Link to the video solution
a)	$\mathcal{L}[e^{-2t}\sin(4t)]$	https://youtu.be/HJtuLoD6XCU?si=dE6k-gZ2-oe6egOq
b)	$\mathcal{L}[e^{2t}\cos 3t]$	https://youtu.be/_CvlSs0T8ok?si=8TKogLDJVSB_csNn

5.4 Second Shifting Theorem

The video in the link below demonstrates what the Unit Step Function means and how to use it in Laplace Transformation. It also shows how to adjust a Unit Step Function for a bounded interval, with worked examples. Towards the end, before introducing the Second Shifting Theorem, the video shows how to use the Unit Step Function to define a piecewise function. The video demonstrates how to apply the Second Shifting Theorem to the following piecewise functions.

https://youtu.be/WqIWCVVlPqw?si=SHFdFuISwvSfAAVp

1. Find the Laplace transformation of

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ t^2, & 1 \le t < 3 \\ 9 & , t \ge 3 \end{cases}$$

2. Find the Laplace transformation of

$$f(t) = \begin{cases} 1, & \le t < \dfrac{\pi}{2} \\ \sin t, & \dfrac{\pi}{2} \le t < \dfrac{3\pi}{2} \\ 0, & t \ge \dfrac{3\pi}{2} \end{cases}$$

3. Find the inverse Laplace transformation of

$$F(s) = \frac{e^{-3s}}{s+4}$$

4. Find the inverse Laplace transformation of

$$F(s) = \frac{e^{-5s}}{s^2+25}$$

The video that comprises in the link below demonstrates how to apply the Second Shifting Theorem to calculate $\mathcal{L}[\cos[3(t-1)]u(t-1)]$.

https://youtu.be/ns2F_WPZxvQ?si=HnDKnv2jfHm6YLCu

5.5 Dirac delta (unit impulse) function

The video in the link below demonstrates the Impulse and Dirac Delta Functions and how to apply them to Laplace transformation. It also demonstrates how to solve the differential equation $y' + y = \delta(t-1), y(0) = 2$ which requires using the unit impulse function.

https://youtu.be/msEeaAlfPcs?si=u1cmVHFL9FZqWxK3

Module 06 - More Laplace Transforms

6.1 Inverse Laplace transformation

The video link below introduces the inverse Laplace transformation and shows its application to essential functions we regularly use.

https://youtu.be/5_lTZyU_BCw?si=L_SEpR4GVxsrlDL6

The videos below demonstrate how to find the inverse Laplace transform for the following problems.

	Problem	Video link
1	$F(s) = \dfrac{1}{s-2}$	https://youtu.be/bMQ8DRTmdkM?si=N-y0oWOQnfROa1Wd
2	$F(s) = \dfrac{1}{2s-1}$	
3	$F(s) = \dfrac{2}{s^5}$	
4	$F(s) = \dfrac{1}{s^2+3}$	https://youtu.be/4a3Ra7kBPO0?si=dshmANwE7CrrXAye
5	$F(s) = \dfrac{5}{s+1} - \dfrac{6}{s^2+4} + \dfrac{1}{s^4}$	
6	$F(s) = \dfrac{s+3}{s^2+4s+13}$	https://youtu.be/TxtXOPt2Ixg?si=SjPgbQozL43-UYql

6.2 First shifting theorem

The video below demonstrates the properties of the First Shifting Theorem and how to apply it to the following problems.

$a)\ f(t) = e^{2t}\sin 4t$ $\qquad$ $b)\ f(t) = \left(t^2 + \frac{1}{6}t^3\right)e^{-2t}$

https://youtu.be/ZSLtHRIASQc?si=OPSfR_Dc4nBK2d6o

The videos linked in the table below show how to apply the First Shifting Theorem to the following problems.

	Problem	Video link
1	$\mathcal{L}[e^{2t}\cos 3t]$	https://youtu.be/_CvlSs0T8ok?si=o_R88BeVVZIVIvro
2	$\mathcal{L}[e^{3t}\cos 2t]$	https://youtu.be/OJ3KrQysI4o?si=JKBZncwn0_3UoM4-
3	$\mathcal{L}[t^2 e^{3t}]$	
4	$\mathcal{L}^{-1}\left[\dfrac{s-2}{s^2-4s+20}\right]$	
5	$\mathcal{L}^{-1}\left[\dfrac{6}{(s-1)^2(s+2)}\right]$	
6	$\mathcal{L}^{-1}\left[\dfrac{2s+4}{s(s^2-6s+13)}\right]$	

7	$\mathcal{L}^{-1}\left[\frac{s+5}{s^2-6s+13}\right]$	https://youtu.be/Wa_ZQyX2uh8?si=rwSzpQk2URzIZ6jq
8	$\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right]$	
9	$\mathcal{L}^{-1}\left[\frac{2s-1}{s^2+4s+29}\right]$	
10	$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+3s+1}\right]$	

6.3 Second Shifting Theorem

The video in the link below demonstrates the properties of the Second Shifting Theorem and how to apply it to the following problems.

Problem	Video link
$\mathcal{L}[(t-3)^2 u(t-3)]$	https://youtu.be/9ZVo781IhRc?si=41SeW9zq-C5O87Bo
$\mathcal{L}[u(t-2)]$	
$\mathcal{L}[93t+1)u(t-3)]$	
$\mathcal{L}\left[\sin t\, u\left(t-\frac{\pi}{2}\right)\right]$	
$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s-1}\right]$	

6.4 Transforms of derivatives

The video below explains the characteristics of the transforms of derivatives method and how to apply it to various functions.

https://youtu.be/zfhyeXbb0d4?si=uXsD8IhNg7IkuwMX

The video in the link below explains the general formulae that can be used for the transforms of derivatives and demonstrates how to apply them to a function with an equation $f(x) = \cos 4t$.

https://youtu.be/2Iye2KsUmF0?si=hiZ9kcoV1-1asQ3d

6.5 Solving initial value problems

The videos in the links below demonstrate how to solve initial value problems.

	Problem	Video link
1	$y'' - y' - 6y = 0$ Given that $y(0) = 2$ and $y'(0) = -1$	https://youtu.be/ZSDr14DGspU?si=c76I_edSzg4C_Zj5
2	$y'' + 4y = \sin 3t$ Given that $y(0) = 0$ and $y'(0) = 0$	
3	$y'' + 6y' + 9y = 0$ Given that $y(0) = -1$ and $y'(0) = 6$	https://youtu.be/6jW7FCyapXE?si=IvYxWfXO19t8hi12

4	$y'' + 4y = t$ Given that $y(0) = 1$ and $y'(0) = 2$	https://youtu.be/AtuYU5VZAl4?si=5_NaNWqD5yVrMebI
5	$y'' - 3y' + 2y = e^{3t}$ Given that $y(0) = 1$ and $y'(0) = 0$	https://youtu.be/HkxzD6BL_Rc?si=xOyBWYZIm8xem6GT
6	$y'' - 2y' + 2y = \cos t$ Given that $y(0) = 1$ and $y'(0) = 0$	https://youtu.be/nSLYEdz_FBU?si=V8zFEoVTZIqkpqrI

Module 07 - Multivariable differential calculus

7.1 Limits

The video link below explains how to calculate limits in multivariable functions.

https://youtu.be/hxrnjxth8sA?si=XklQgh11qf2Mdz3i

The video below demonstrates how to evaluate limits for the following multivariable functions at given points.

https://youtu.be/E1lMMBpz8YM?si=ScpiNRC_HrmDM19V

$a)\ \lim_{(x,y)\to(1,2)} x^3 + 3xy - 2y^2$ $\quad b)\ \lim_{(x,y)\to(5,5)} \frac{x^2 - y^2}{x - y}$ $\quad c)\ \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$

$d)\ \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + y^4}$ $\quad e)\ \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 4} - 2}$ $\quad f)\ \lim_{(x,y,z)\to(0,0,0)} \frac{y^2 + 3xz^2}{x^2 + 2y^2 + z^4}$

7.2 Partial derivatives

The video below explains partial derivatives and their use in modelling and mathematical applications.

https://youtu.be/ly4S0oi3Yz8?si=9ij5eiwJg2VvPqGg

7.2.1 First-order partial derivatives

The video below explains partial derivatives and demonstrates how to find first-order partial derivatives for a simple question.

https://youtu.be/d15V7ZdVYyE?si=b45D6ksIvOAceEXL

The video at the link below shows how to find partial derivatives for the following questions. At the end of the video, it shows a few questions on how to find second-order partial derivatives.

https://youtu.be/JAf_aSIJryg?si=pA2LdB4BsEURgHyZ

1. $a)\ f(x,y) = 7x^2 - x^3y^4 + 5x^4y^3$ $\quad b)\ f(x,y) = 3x^2y^4 - 5x^7 + 4y^8$ $\quad c)\ z = e^{x^2y^3}$

   $d)\ z = x^4e^{y^5}$ $\quad d)\ f(x,y) = \ln(x^2 + y^2)$ $\quad e)\ z = \ln\left(\frac{x^2}{y}\right)$

   $f)\ f(x,y) = \sqrt{x^2 + y^2}$ $\quad g)\ z = \sin(x^3y^5)$

2. Find the partial derivative values at the point with coordinates $(1,2)$ for the function with the equation

$$f(x,y) = 2x^3y^2 + 5y^3 + 4x^2$$

3. Find the partial derivative values at the point with coordinates $P(2,0)$ for the function with the equation

$$f(x,y) = x^3e^{4xy^2}$$

4. Find the partial derivative for the function with the equation,

$$f(x,y) = \frac{3y^5}{x^3 + y^3}$$

5. Find the slope of the surface in the x and y directions at the point with coordinates (3,2)

$$f(x,y) = x^2y^3$$

6. Find the partial derivative with respect to x, y and z;

$$f(x,y,z) = x^5y^2z^4$$

$$w = x^3(\ln zy)e^{x^2y^3z^4}$$

7. Find the partial derivative with respect to x, y and z and then evaluate at (1,2,3)

$$f(x,y,z) = 4xy^2 + 3zx^3 + 5yz^4$$

7.2.2 Second-order partial derivatives

The video linked below explains how to find second-order partial derivatives and demonstrates it with the following questions. The video also explains mixed partial derivatives.

https://youtu.be/Xn6LjHKMrqA?si=V0G560mo6LLgEDxX

$a)\ f(x,y) = 2x^4 - 5x^3y^3 - y^4$

$b)\ f(x,y) = xe^{xy^2}$

$c)\ f(x,y) = e^{xy^2}(xy^2 + 1)$

The video contained in the link below demonstrates how to find all the second-order derivatives for;

$$z = x^3y^5$$

https://youtu.be/s7XVLH3MA-s?si=s7HgdJwrkOgJA4Ca

7.3 Chain Rule in multivariable calculus

The video below explains the fundamental idea behind the Chain Rule in multivariable differentiation. The video also demonstrates applying the Chain Rule for the multivariable function with the equation $f(x,y) = x^2y, x(t) = 2t + 1, y(t) = t^3$

https://youtu.be/9yCtWfI_Vjg?si=TEInLZqP5RdcE7b6

The video in the link below demonstrates how to apply the Chain Rule to the following problems.

https://youtu.be/XipB_uEexF0?si=MSnT67zplqlp0DSx

1. Find $\frac{dz}{dt}$ for $z = x^3y + 3x^2y^3$, given that $x = 4 + t^2$ and $y = 5t^3$
2. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = 4x^3y^5$, given that $x = 3s^2t$, and $y = 2s^3t^2$
3. Find $\frac{\partial u}{\partial t}$, given that $u = x^{2y} + y^3z^2, x = rs^2t, y = r^2st^3$ and $z = rst$

7.4 Applications of partial derivatives

The video linked below demonstrates the applications of partial derivatives in various contexts.

https://youtu.be/w4zHG9g3Bwc?si=CNmtAL2WVFx8ojXF

1. Given that

$$y = \frac{ws^3}{d^4}$$

Find the percentage change in y when w increases by 2%, s decreases by 3% and d increases by 1%.

2. The power consumed by a resistor is given by the following relationship,

$$P = \frac{V^2}{R}\ Watts$$

Determine the approximate change in power when the voltage, V changes by 3%, and the resistance R decreases by 0.4%. The original values of V and R are 40 Volts and 8 Ohms, respectively.

3. The current flowing in an electric circuit is given by Ohm's law, $i = \frac{V}{R}, Amps$. Calculate the approximate change in the current if the voltage decreases from $120V$ to $118V$ and the resistance increases from Ω to 200.4Ω.

The videos in the playlist below demonstrate how partial differentiation applies in various contexts, such as one-dimensional heat equations, wave equations, two-dimensional heat equations, and telephone and radio equations.

https://youtu.be/RtVE2Gt-KQ4?si=lfPyRtzT5OGHoFy9

Module 08 – Directional derivatives, gradient vectors, equations of lines and planes

8.1 Introduction to directional derivatives

The video linked below demonstrates directional derivatives and how to find the slope in any direction using partial derivatives.

https://youtu.be/GJODOGq7cAY?si=U1ySxWyqxh1-kw_5

The video below demonstrates how to find the directional derivative in the given directions and at the given points.

https://youtu.be/CnVes9TdnPo?si=3On-X9jd7fX2Jvw4

1. Given that $f(x,y) = 4x^3 - 3xy^2$
   a) Find the directional derivative in the direction given by the angle $\theta = \frac{\pi}{3}$
   b) Evaluate the directional derivative at the point $(1,2)$.

2. Given that $f(x,y,z) = 3xy^3 - 2xz^2$
   a) Find the directional derivative in the direction of the vector $v = 2i - 3j + 6k$.
   b) Evaluate the directional derivative at the point $(3, 1, -2)$.

3. Given that $f(x,y) = xy^2 + 2x^3$ in the direction of the angle $\theta = \frac{\pi}{6}$
   a) Evaluate the gradient vector at the point $(1, -3)$.
   b) Find and evaluate the directional derivative at the point $(1, -3)$.

4. Given that $f(x,y,z) = 2xz^3 - 3yz^2$
   a) Find the gradient of f at $P(2,3,1)$.
   b) Find the directional derivative of the direction of f at the same point in the direction of $v = 4i - j + 8k$

8.2 Applications of Directional Derivatives

The video in the link below demonstrates how to solve the following applications of directional derivatives.

https://youtu.be/iof34zn5sqw?si=LfHSPQCEm1tBYE1X

The video at the link below provides a brief introduction to directional derivatives and the gradient vector. It also shows how to solve the following problems.

https://youtu.be/iof34zn5sqw?si=6_vFMx_vuRFYU1l4

1. Find the directional derivative of $f(x,y) = \frac{x}{y}$ at $(6, -2)$ in the direction of $v = -i + 3j$

2. Find the directions in which the directional derivative of $f(x,y) - x^2 + \sin xy$ at $(1,0)$ has the value 1.

3. Find the directional derivative of $f(x,y) = x^2 + 6xy - 3y^2$ at the point $P(1, -1)$ in the direction of the unit vector whose direction is given by the angle $\frac{\pi}{3}$.

4. Find the directional derivative of $f(x,y,z) = z^3 - x^2y$ at $(1,6,2)$ in the direction of $v = 3i + 4j + 12k$.

5. The temperature at a point (x, y, z) is given by $T(x, y, z) = 2xy - yz$, where T is measured in °C and x, y and z in meters.
   a) Find the rate of change in temperature at $P(2, -1,2)$ in the direction towards the point $(3, -3,3)$
   b) In which direction does the temperature increase fastest at P? What is this rate of increase?
   c) At point P, is there a direction in which the rate of change in temperature is -5 °C$/m$? If so, where? If not, why not?

6. Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.01x^2 - 0.02y^2$$

where x, y, z are measured in meters, and the positive x-axis points are due east, while the positive y-axis points are due north. Suppose you are at the point with coordinates $(50,80,847)$.

a. If you walk due south, will you start to ascend or descend? At what rate?

b. If you walk northwest, will you start to ascend or descend? At what rate?

c. In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

7. Find the tangent plane to $x^2 - 2y^2 + z^2 = 3$ at $(-1, 1, -2)$
8. Find the points on $x^2 + 2y^2 + 3z^2 = 1$, where the tangent plane is $3x - y + 3z = 1$

8.3 Tangent planes and normal lines

The video below explains how to find the tangent plane and the normal vector. It demonstrates how to find the tangent plane and normal line to the level surfaces below.

https://youtu.be/o9EGq5qpons?si=nbwssyscheZL-JxA

1. Find the tangent plane and the normal line to the level surface at $(1,2,4)$ for the function with equation $f(x, y, z) = x^2 + y^2 + z - 9$.

2. Find the tangent plane and the normal line to the level surface at $(1,1)$ for the function with equation $f(x, y) = -x^4 + x^2 - y^2$.

Module 09 – Extreme values, critical points and Lagrange multipliers

9.1 Maximum and minimum in multivariable function

The video below explains how to find the maximum and minimum points of a multivariable function using the first and second derivative tests, including how to determine saddle points.

https://youtu.be/_Ffcr98c7EE?si=ztONwOoTLCY_62YO

The video linked below demonstrates how to determine the absolute maximum and minimum of the following functions.

Find the absolute minimum and maximum values of the functions in the table below.

Problem	Video link
$f(x,y) = x^2 + y^2 + 4x - 6y$ on the region $D = \{(x,y) \mid x^2 + y^2 \leq 16\}$	https://youtu.be/_5gB5sC0Ra4?si=PsKzqiUT78ck5sHw
$f(x,y) = x^2 - 2xy + 4y$ on the region $D = \{(x,y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$	https://youtu.be/Hg38kfK5w4E?si=WSIrmKKeDoy-cS02

9.2 Lagrange multipliers for finding the extreme value of a function

The video below demonstrates Lagrange multipliers for finding the extreme value of a function and its geometric interpretation, along with examples.

https://youtu.be/8mjcnxGMwFo?si=6wdstJxvksGVRCb5

The video below demonstrates how to apply Lagrange multipliers to find the maximum and minimum for the following problems.

https://youtu.be/5-CUqogfPLY?si=iTqt_QjTTqzxuTs0

1. Given that $f(x,y,z) = 3x^2 + y^2 - 2z^2$ and $3x + 2y - 8z = -50$, use Lagrange multipliers to find any maximum or minimum values.
2. Given that $f(x,y,z) = 4x + 2y + 6z$ and $x^2 + y^2 + z^2 = 14$, use Lagrange multipliers to find any maximum or minimum values.
3. Given that $f(x,y,z) = xyz$ and $x^2 + 4y^2 + 16z^2 = 48$, use Lagrange multipliers to find any maximum or minimum values.
4. Given that $f(x,y,z) = 4x + 6y - 2z$, $2x + y + 5z = 1$ and $y^2 + 2z^2 = 22$, use Lagrange multipliers to find any maximum or minimum values.

9.3 Applications of maximum and minimum problems

1. Two types of TVs are produced in two factories, A and B. Suppose type "X" TVs are produced at factory A, and type "Y" TVs are produced at factory B. The production cost C, of making TVs is given by $C(x,y) = 6x^2 + 12y^2$

   If you must produce 90 TV sets from both types, how many will be produced in each type?

   https://youtu.be/ry9cgNx1QV8?si=02G-6idCoQW1R5az

2. Please note that the video linked below demonstrates solutions for four questions. However, only the final question addresses modelling with Lagrange multipliers, and the last four

minutes of this video show the model solution to that problem. The question is outlined below, followed by the link.

https://youtu.be/x6j6yFzTUgU?si=IajPupxTV_qRR7tK

In economics, a Cobb-Douglas equation is used to compute total production Y as a function of labour input L and capital input K. One such equation is $Y = 32L^{0.6}K^{0.4}$.

Maximise production subject to the constraint $4L + 2K = 50$.

Module 10 - Double integrals in rectangular coordinates

10.1 Introduction to double integrals

The video at the link below demonstrates how to set up double integrals and the idea behind using them to generate the volume of a solid.

https://youtu.be/gifQWtTWqEY?si=XTaTq_rj9WOryowT

10.2 Fubini's theorem 1

The videos in the links below demonstrate how to apply Fubini's theorem 1 to the problems listed.

Problem		Video link
1	$\int_0^2 \int_0^{\frac{\pi}{2}} x \sin y \ dydx$	https://youtu.be/kxho042-R6k?si=T7cqEYiy-gg1eGKJ
2a)	$\int_0^2 \int_1^3 xy^2 dydx$	https://youtu.be/BJ_0FURo9RE?si=1Tnm8fN8mFZHpw_o
2b)	$\int_0^1 \int_0^2 2y - 3x^2y^2 dydx$	
2c)	$\int_0^1 \int_0^{2y} 4 + 2x - y^2 dxdy$	
2d)	$\int_0^3 \int_0^2 (4x + y)^3 dxdy$	
2e)	$\int_0^3 \int_0^2 xye^{x^2y} dydx$	
3	$\int_0^{\pi} \int_1^2 y\sin(xy) dxdy$	https://youtu.be/hcBiYZuST7U?si=DUZhQPG_BbL5toXM

10.3 Fubini's theorem 2

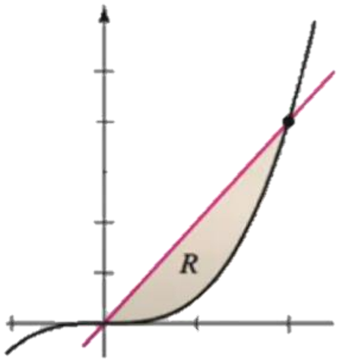
The videos in the links below demonstrate how to apply Fubini's theorem 2 to the problems listed.

https://youtu.be/m5HD9ZUGlr8?si=ZNaARKEyQVileYiq

1. Evaluate the area bounded by the region $y = 2x^2$ and $y = 1 + x^2$

   https://youtu.be/fJ5bA7Dod_o?si=HpYawH5J6SP46-hr

2. Consider the region R. Set up a generic integral and rewrite your answer in the order of integration.



3. Consider the triangular region R with vertices (0,0), (4,2) and (6,0). Set up a generic integral, then rewrite your integral in the other order.

4. The integral below can only be evaluated by reversing the order of integration. Sketch the region in question, rewrite the integral with the reversed order, and finally evaluate the new integral.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\, dy dx$$

10.4 Finding the average value of an integrable function of two variables

The videos in the links below demonstrate how to find the average value of the integrals listed in the table.

<table>
<tr><th colspan="2">Problem</th><th>Video link</th></tr>
<tr><td>a)</td><td>$f(x,y) = 6x^2y$
$R = \{(x,y)\mid -1 \le x \le 2, 0 \le y \le 2\}$</td><td rowspan="2">https://youtu.be/L93gt4htCvU?si=OexA42Uhs_SEQ2-W</td></tr>
<tr><td>b)</td><td>$f(x,y) = \dfrac{xy^2}{1+x^2}$
$R = \{(x,y)\mid 0 \le x \le 1, -3 \le y \le 3\}$</td></tr>
<tr><td>c)</td><td>$f(x,y) = xy$, D is the triangle with vertices $(0,0), (1,0)$ and $(1,3)$</td><td rowspan="2">https://youtu.be/aEwYMgXvf9U?si=gnttYa0IsnObP5EH</td></tr>
<tr><td>d)</td><td>$f(x,y) = x\cos y$
D is the region bounded by the curves
$y = x^2, y = 0, x = 1$</td></tr>
</table>

10.5 Applications of double integrals

The videos in the playlist below demonstrate various applications of double integrals, including finding the mass, centre of mass, and Jacobian.

https://www.youtube.com/watch?v=2y8dJzd4FPw&list=PLROOIV7hGpZjzJYKw4aOc6POn_ga2EVnx

Module 11 - Double integrals in polar coordinates

The videos in the playlist below demonstrate various applications of double integrals in polar coordinates. The first video introduces polar-coordinate area and then demonstrates how to use double integrals to find the volume of any solid, with examples. The remaining videos in the playlist then demonstrate various examples of finding the volume of solids in polar form.

https://www.youtube.com/watch?v=5ixPVUNBi88&list=PLROOIV7hGpZhPtuAhl-IXsQo4IqHxH0f4

Module 12 – Triple integrals in rectangular coordinates

The videos in the playlist below demonstrate the triple integral. The first video introduces the idea behind the triple integral and then demonstrates how to apply it to evaluate the volume of various objects created by curves.

https://youtube.com/playlist?list=PLX2gX-ftPVXXbQFICCORLptPh896DuPQ2&si=WwDw1rwF6uk7IDsK