MATH1016 Mathematics for Engineers 1

additional resources created by

Mathematics Education Support Hub (MESH)

Contents

1 How to Use This Resource?

Purpose of This Resource

This additional study material is designed to help you better understand the content of **MATH1016 Mathematics for Engineers 1**. It provides a guided, hands-on approach to mastering key mathematical concepts and serves as a supplement to your lectures and tutorials.

Research has shown that although many students seek additional support online, the resources they find often do not correspond to the content or learning outcomes of their course. This can lead to confusion, frustration, and unnecessary stress. To address this, we have created a resource aligned with the MATH1016 Mathematics for Engineers 1, covering the required depth and breadth of the course.

What You'll Find in This Resource

- **Step-by-step video solutions** to selected problems from each topic
- **Worked examples** demonstrating how to approach different types of questions
- **Practice questions with answers** to reinforce your understanding
- **Modelling and problem-solving examples**, including visual aids where applicable

How to Use It Effectively

- Don't just watch passively; Pause after a few examples and try similar problems on your own.
- Compare your working with the solutions provided to identify any gaps in your understanding.
- Pay attention to mathematical terminologies; this is especially useful if English is not your first language. Familiarising yourself with these terms early will strengthen your comprehension.

Special Focus on Problem Solving

According to research, students often struggle with modelling, problem-solving, and applying maths to real-world situations. To help with this, the resource includes:

- A variety of problem-solving questions with detailed explanations
- Tips such as:
  - **Reading questions carefully** (more than once if needed)
  - **Sketching diagrams** to visualise problems
  - **Taking time to fully understand** the question before attempting a solution

With regular practice and the right strategies, you can build the confidence and skills needed to succeed in your course.

Need More Help?

If you require additional support, the **MESH** (Mathematics Education Support Hub) team is here to assist you. You can:

- Attend online or in-person support sessions
- Join workshops designed to help you prepare for class tests

For more information, contact MESH at: mesh@westernsydney.edu.au

The following QR codes direct you to the Mathematical Terminologies and MESH services.

Mathematical Terminologies

MESH Services

2 Functions and Inverse Functions

Objectives

This chapter covers the following topics related to functions. After successful completion of this section, you will be able to:

- Define a function and determine the domain and range of functions.
- Define what a one-to-one function is and calculate the inverses of such functions.
- Give examples of common functions and their inverses.
- Manipulate expressions and equations involving logarithms and exponentials.
- Define and graph logarithmic and exponential functions.
- Manipulate expressions and equations, and graph functions such as trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions.

2.1 Introduction to Functions: Finding the Domain and Range of a Given Function

The following video provides an introduction to finding the domain and range of a given function for various graphs.

https://www.youtube.com/watch?v=KirGQOwjBVI

2.2 Domain and Range of a Function

1) The following video demonstrates how to find the domain for the functions listed below.

https://www.youtube.com/watch?v=grpnc9ME4fU

a) $f(x) = \dfrac{x+3}{x-7}$

b) $f(x) = \dfrac{2x+1}{x^2-16}$

c) $f(x) = \dfrac{x^2}{x^2+9}$

e) $f(x) = \sqrt{x-9}$

f) $f(x) = \sqrt{x^2+5x-21}$

g) $f(x) = \sqrt[3]{-x+5}$

i) $f(x) = \dfrac{\sqrt{x+5}}{x^2-5}$

j) $f(x) = \dfrac{3x-4}{\sqrt{x+9}}$

k) $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x^2-36}}$

2) The following video demonstrates how to find the domain and range for the following functions.

https://www.youtube.com/watch?v=Si2vmzUWfJE

i) **Linear functions**

a) $f(x) = x + 3$

ii) **Quadratic functions**

a) $f(x) = x^2$ b) $f(x) = x^2 - 3$ c) $f(x) = 4 - x^2$

d) $f(x) = x^2 - 4x + 5$

iii) **Cubic functions**

a) $f(x) = x^3$ b) $f(x) = x^3 + 5x^2 - 8$ c) $f(x) = 4 - x^2$

iv) **Absolute (modulus) functions**

a) $f(x) = |x|$ b) $f(x) = |x| + 3$ c) $f(x) = |x - 2| - 3$

d) $f(x) = 2 - |x - 3|$

v) **Radical functions (Functions with roots)**

a) $f(x) = \sqrt{x}$ b) $f(x) = -\sqrt{x}$ c) $f(x) = \sqrt{-x}$

d) $f(x) = -\sqrt{-x}$ e) $f(x) = -\sqrt{x - 3} + 4$

2.3 Functions and Relations

The following video demonstrates the distinction between function and relationship. It also explains the types of functions, such as one-to-one and many-to-one.

https://youtu.be/6u9Qh1Vi_Qw?si=94J-EPuk4xGuX9BK

2.4 Functions and Their Inverses

The following video demonstrates the relationship between functions and their inverses, along with the following worked examples.

https://www.youtube.com/watch?v=1aBkyMVvlsU

i) **Linear functions**

a) $f(x) = 7x + 2$ b)$f(x) = 8x - 5$ c) $f(x) = \frac{x}{5} + 1$

ii) **Quadratic functions**

a) $f(x) = x^2 + 4$

iii) **Radical functions**

a) $f(x) = \sqrt{2x^3 + 58}$

iv) **Inverse composite functions**

a) $f(x) = x + 5$ and $g(x) = x^2 - 2$, workout $(gf)^{-1}(x)$

2.5 Exponential and Logarithmic Functions

The videos included in the links below demonstrate expressions and equations involving logarithms and exponents. They also define and graph logarithmic and exponential functions, as well as find their domain and range.

i) The link below demonstrates how to find the inverse of the following logarithmic functions.

https://www.youtube.com/watch?v=hNsvGz7JPJQ

a) $f(x) = \log_2(x - 3) - 5$

b) $f(x) = 2\log_4(x - 5) + 12$

c) $f(x) = \ln(x - 3) + 4$

d) $f(x) = 5 - \ln(6 - \frac{1}{3}x)$

ii) The link below demonstrates how to find the inverse of the following exponential functions

https://www.youtube.com/watch?v=ihkYZOwPetM

a) $f(x) = 3^x + 5$

b) $f(x) = 2(4^x) - 10$

c) $f(x) = 4e^x - 20$

d) $f(x) = 8 - 2e^{5x}$

2.6 Trigonometric and Hyperbolic Functions: Their Graphs, Inverses, Equations, and Identities

i) The following video demonstrates the relationship between the standard trigonometry functions and their inverse.

https://www.youtube.com/watch?v=YXWKpgmLgHk

ii) The following videos demonstrate sketching the inverse trigonometric functions listed below.

Function	Link to access the video
$y = sin^{-1}x$	https://youtu.be/ZM52fcPp1yw?si=d4FzuWfG3UTb1ny3
$y = cos^{-1}x$	https://youtu.be/5o2lqjX-p_Q?si=VG08XxSQGOoVXLAa
$y = tan^{-1}x$	https://youtu.be/1A-IbjIIy_I?si=9p4zYMsjjIWgHTKI

iii) The following videos demonstrate various aspects of hyperbolic functions, including graphing and solving identities.

Description	Link to access the video
Introduction	https://youtu.be/vVcEePflDUA?si=7hk8GLRdvBE94uTf
Graphs	https://youtu.be/CJzYpiZZD07?si=mAAD21lP8HuSelBY
Graphs of inverse	https://youtu.be/TaVdqXsSksA?si=Rh6P7hRBlXen9DHe
Identities	https://youtu.be/G1xHIXmz5HM?si=YTYvbPAfsBXMJw40
Solving equations	https://youtu.be/gIZwP5r08Rk?si=xLfx5cQJoW5uxr6

3 Differential Calculus

Objectives

This chapter covers the following topics related to differential calculus. After successful completion of this section, you will be able to:

- Calculate the limit of a function.
- Determine whether a function is continuous or not at a given point.
- Define what is meant by the derivative of a function.
- Differentiate various functions using the appropriate techniques and rules.
- Use L'Hopital's Rule to find derivatives.
- Calculate the equation of a tangent line and a normal.
- Use differentials to estimate the change.
- Sketch curves using the first and second derivatives.
- Find vertical and horizontal asymptotes.
- Use differential calculus to solve related rate problems.

3.1 Introduction to Limits

The following video provides an introduction to limits, helping you familiarise yourself with limits and how to find the limits for various functions.

https://www.youtube.com/watch?v=riXcZT2ICjA

3.2 Finding the Limit of a Given Function

The following video demonstrates how to find the domain for the functions listed below.

https://www.youtube.com/watch?v=YNstP0ESndU&t=758s

a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$ b) $\lim\limits_{x\to 2}(x^2+2x-4)$ c) $\lim\limits_{x\to 3} \dfrac{x^3-27}{x-3}$

d) $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ e) $\lim\limits_{x\to 4} \dfrac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}$

3.3 Checking the Continuity of a Function at a Given Point

The following video demonstrates how to find the continuity check of a function at a given point. This process is explained using the 3-Step Continuity Test for the functions listed below.

https://youtu.be/WT7oxiiFYt8?si=a2rwCgsuWi7MguMs

$$a) f(x) = \begin{cases} \sqrt{x+2}, & x < 2 \\ x^2 - 2, & 2 \leq x < 3 \\ 2x + 5, & x \leq 3 \end{cases} \qquad b) f(x) = \begin{cases} 2x + 5, & x < -1 \\ x^2 + 2, & x > -1 \\ 5, & x = 1 \end{cases}$$

3.4 Derivative of a Function

The video below demonstrates the concept of derivatives and how the limit approach is applied to find the derivative of a function, including the concept of rate of change, through specific examples.

https://youtu.be/wUZcIYx-7a4?si=-2oKH5YW-DpHM6N5

i) Derivative of polynomial functions

The video below shows how to find the derivative of the polynomial functions listed below.

https://youtu.be/8Sv6CNuNwqo?si=l27V8TE15zc7NX0K

a) $f(x) = x^3 - 5x^2 + 7x - 4$ b) $f(x) = 4x^5 - 6x^3 + 8x^2 - 9$

c) $f(x) = 7x(2x - x^3)$ d) $f(x) = (3x + 2)^2$

e) $f(x) = \dfrac{4x^5 - 5x^4 + 2x^3}{x^2}$

ii) Derivative of implicit functions

The following video demonstrates how to find the derivative of the implicit functions listed below.

https://youtu.be/xbviQHhU1rA?si=6bepCQjQn361XoiT

a) $x^2 + y^2 = 100$ b) $x^3 + 4xy + y^2 = 13$ c) $5 - x^2 = \sin(xy^2)$

iii) Derivative of exponential functions

The video below shows how to find the derivative of the exponential functions listed below.

https://www.youtube.com/watch?v=yg_497u6JnA

a) $\dfrac{d}{dx}\left[e^{x}\right]$ b) $\dfrac{d}{dx}\left[e^{5x+3}\right]$ c) $\dfrac{d}{dx}\left[e^{x^2}\right]$ d) $\dfrac{d}{dx}\left[e^{x^3+8x}\right]$

e) $\dfrac{d}{dx}\left[3^{x}\right]$ f) $\dfrac{d}{dx}\left[7^{2x-5}\right]$ g) $\dfrac{d}{dx}\left[9^{x^3}\right]$ h) $\dfrac{d}{dx}\left[5^{2x-x^2}\right]$

i) $\dfrac{d}{dx}\left[e^{\sin x}\right]$ j) $\dfrac{d}{dx}\left[4^{\tan x}\right]$ k) $\dfrac{d}{dx}\left[x^3e^{4x}\right]$ l) $\dfrac{d}{dx}\left[\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right]$

iv) Derivative of logarithmic functions

The video below shows how to find the derivative of the logarithmic functions listed below.

https://www.youtube.com/watch?v=Dp9sgIvaKPk

a) $\dfrac{d}{dx}\left(\ln x\right)$ b) $\dfrac{d}{dx}\left(\ln x^2\right)$ c) $\dfrac{d}{dx}\left(\ln x^3\right)$

d) $\dfrac{d}{dx}\left(\ln(x+5)\right)$ e) $\dfrac{d}{dx}\left(\ln(x^2+4)\right)$ f) $\dfrac{d}{dx}\left(\ln(5+7x-x^3)\right)$

g) $\dfrac{d}{dx}\left(\ln(\sin x)\right)$ h) $\dfrac{d}{dx}\left(\ln\left(\sqrt[7]{x}\right)\right)$ i) $\dfrac{d}{dx}\left(\sqrt[3]{\ln x}\right)$

j) $\dfrac{d}{dx}\left(\log_3(x)\right)$ k) $\dfrac{d}{dx}\left(\log_7(5-2x)\right)$ l) $\dfrac{d}{dx}\left(\log_2(3x-x^4)\right)$

v) Derivative of trigonometric functions

The video below shows how to find the derivative of the trigonometric functions listed below.

https://www.youtube.com/watch?v=eEn6n60dv1Q

a) $f(x)=\sin(2x)$ b) $y=\cos(x^2)$ c) $y=x^2\cos x$

d) $f(x)=\cos^2 x$ e) $y=5\tan(3x)$ f) $y=\dfrac{\sin x}{x}$

g) $y=\dfrac{x^2\tan x}{\sin(2x)}$ h) $y=\sin^5(x^3+4x^2-2)$

vi) Derivative of hyperbolic functions

The video below shows how to find the derivative of the hyperbolic functions listed below.

https://www.youtube.com/watch?v=Q6-QZxUDfE0

a) $\dfrac{d}{dx}\left[\sinh x\right]$ b) $\dfrac{d}{dx}\left[\sinh(4x)\right]$ c) $\dfrac{d}{dx}\left[\cosh(x^3)\right]$

d) $\dfrac{d}{dx}\left[(x^4)\right]$ e) $\dfrac{d}{dx}\left[x^2\sinh x\right]$ f) $\dfrac{d}{dx}\left[\ln(\sinh(x))\right]$

vii) Derivative of inverse trigonometric functions

The following video demonstrates how to find the derivative of the inverse trigonometric functions listed below.

https://youtu.be/J2HmfIvDVCo?si=4O3OJ-dDhsq6rJTi

a) $f(x) = \sin^{-1}(5x)$ b) $y = \sin^{-1}(e^{6t})$

c) $g(x) = \sec^{-1}(e^x)$ d) $y = \tan^{-1}(x^2)$

e) $y = (\tan^{-1} x)^2$ f) $y = 2x^4 \sin^{-1}(x)$

g) $y = \tan^{-1}\left(\sqrt{5x}\right)$ h) $y = x\sin^{-1} x + \sqrt{1-x^2}$

i) $h(x) = (\arcsin x)\ln x$ j) $y = -\cos^{-1}\left(\dfrac{6x+3}{9}\right)$

viii) Derivative of inverse hyperbolic functions

The following video demonstrates how to find the derivative of the inverse hyperbolic functions listed below.

https://youtu.be/NQGiOuohNqI?si=gLU6AXjUXk2GcvSu

a) $\dfrac{d}{dx}\left(\sinh^{-1}(4x)\right)$ b) $\dfrac{d}{dx}\left(\cosh^{-1}(x^3)\right)$

c) $\dfrac{d}{dx}\left(\tanh^{-1}(\sqrt{x})\right)$ d) $\dfrac{d}{dx}\left[\csc^{-1}(x)\right]^3$

e) Find $\dfrac{dy}{dx}$ for $y = 6x\sinh^{-1}(3x) - 2\sqrt{1+9x^2}$

3.5 L'Hopital's Rule

The video linked below demonstrates how to apply the L'Hôpital rule to find the limit of the following problems.

https://youtu.be/Gh48aOvWcxw?si=XM-H4kZ10GVwUZaZ

(a) $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}$ (b) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$ (c) $\lim\limits_{x\to 0}\dfrac{\sin(7x)}{\sin(4x)}$

(d) $\lim\limits_{x\to 0}\dfrac{\sin(8x)}{3x}$ (e) $\lim\limits_{x\to\infty} x\ln x$ (f) $\lim\limits_{x\to\infty}\dfrac{1}{x^x}$

(g) $\lim\limits_{x\to 0}[1-2x]^{\frac{1}{x}}$

3.6 Tangents and Normals

i) The video in the link below demonstrates the concept of finding tangent and normal equations using the curve with the equation $y = x^2 + 1$ at $x = 3$.

https://youtu.be/9pWiu7VxXP4?si=HKq65zSEUipYqOBb

ii) The video on the link below demonstrates how to find the tangent equation for curves with equations.

https://youtu.be/N_FM6aON2z8?si=mMODVCsuF6BSBm6-

(a) $y = x^2 - x + 3$ at $(5,\ 15)$

(b) $y = x^3 - 2x^2 + 10$ at $(1,\ 9)$

(c) $y = x(x^2 - 5)$ at $x = 3$

3.7 Related Rate Word Problems

The videos in the links below demonstrate how to solve the following examples related to related rates.

Example 1: The side of a cube is increasing at the rate of $5\,\text{cm}\,\text{s}^{-1}$. Find the rate of increase of the volume when the length of a side is 3 cm.

Solution: https://youtu.be/BucbEKBz2dY?si=Q0s_tmTapdSRKeZJ

Example 2: A water tank (inverted cone) with a radius of 2 m and a height of 4 m. If water is pumped in at a rate of $2\,\text{m}^3$ per minute, find the rate at which the water level rises when the water is 3 m deep.

Solution: https://youtu.be/iK625WaKomM?si=iVpleSRf5lphGh85

3.8 Graph Sketching by Investigating Stationary Points

The following videos demonstrate how to sketch curves by examining stationary points and their nature, as well as identifying intervals where the function is increasing or decreasing, for the examples listed below.

Example 1: Sketch the curve with the equation $y = x^3 + 3x^2 - 9x - 4$

Solution: https://youtu.be/H-XDX7T0ADw?si=Q1lR6Pc4-aotcO4f

Example 2: Sketch the curve with the equation $y = \dfrac{x^2}{x^2 - 1}$

Solution: https://youtu.be/57bjrMD9qbq?si=DAmhJczbZfmaMQL

4 Vectors

Objectives

This chapter covers the following topics related to vectors. After successful completion of this section, you will be able to:

- Identify and distinguish between scalar and vector quantities.
- Define what is meant by a position vector.
- Use the laws of vector algebra to simplify expressions involving vectors and scalars.
- Find the Cartesian co-ordinate form of position vectors in 2 and 3 dimensions.
- Calculate the dot product of two vectors.
- Find the angle between two vectors.
- Calculate the cross product of two vectors.
- Find the equations of lines and planes.

4.1 Definitions, Properties and Characteristics of Vectors

The video in the following link demonstrates the properties of vectors, including distinguishing between vector quantities and scalar quantities, the notations used to represent vectors, methods for describing vectors, the magnitude of a vector, and visualising vectors in space.

https://youtu.be/2_21erD-nBg?si=Y-bjo5eON2VnQku4

Note: Expressing a vector can be done in multiple ways. For example, in this video, vectors are noted as $A = \langle 4, 2 \rangle$, which represents the position vector of A. Sometimes this is also expressed as $\vec{\text{OA}} = 2\mathbf{i} + 3\mathbf{j}$.

The video also demonstrates how to find solutions to the following questions.

1. The vector $\mathbf{V}$ has an initial point $(1, -2)$ and terminal point $(5, 1)$.
   (a) Find the component from the vector $\mathbf{V}$
   (b) Determine the magnitude of the vector $\mathbf{V}$
2. The vector $\mathbf{V}$ has an initial point $A(-4, 1)$ and a terminal point $B(8, 6)$. The vector $\mathbf{U}$ has an initial point $C(-7, -15)$ and a terminal point $D(3, 9)$. Determine whether the two vectors are equivalent.

After finding the answers to questions 1 and 2, the video extends to demonstrate **adding** and **subtracting** vectors. Then it demonstrates the answer to the following question.

3. If $A = \langle 4, -2 \rangle$ and $A = \langle -3, 5 \rangle$, find **2A + 3B** and **5A - 4B**.

After question 3, the video extends to demonstrate a few more key concepts, like **position vector** and **unit vector**. At the end, the video demonstrates how to find answers to the following questions.

4. Find the unit vector in the direction of $\mathbf{v}$, where $A = \langle 4, -3 \rangle$.
5. Write the vector $\mathbf{V}$ as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ given that vector $\mathbf{V}$ has a magnitude of 16 and an angle of 30° with the positive X axis.
6. Find the magnitude of the vector for $\mathbf{v} = 3\mathbf{i} - 8\mathbf{j}$, and determine the angle that it makes with the positive X-axis
7. Find the resultant force of two vectors $\mathbf{F}_1 = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{F}_2 = -2\mathbf{i} - 8\mathbf{j}$

4.2 Vector Resolution

The video comprises in the following link demonstrates how to find the **magnitude** and the **T**he video in the following link demonstrates how to find the **magnitude** and **direction** of the **resultant** of two forces (vectors) by **resolving** the vectors for the example outlined below.

https://youtu.be/dC4cudpxVQw?si=PKPbo0QUqlFZe87O

Example: Find the component of each vector.

a) Force 300 N at 30° about the X axis.

b) Force 200 N at 210° below the X axis.

4.3 The Dot Product (Scalar Product) and Finding the Angle Between Two Vectors

The video in the link below demonstrates the dot product calculation and how to find the angle between two vectors for the following problems.

https://youtu.be/VzX8KJKFhlM?si=7Vs3qHURq1M0PHYB

Questions 1 to 5 demonstrate how to find the dot product of two vectors.

1. $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j}$
2. i) $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$

   ii) $\mathbf{a} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$
3. Given that vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a} = 5\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 7\mathbf{i} + 8\mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j}$. Find,

i) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

ii) $(3\mathbf{a}) \cdot \mathbf{b}$

iii) $4(\mathbf{a} \cdot \mathbf{b})$

4. i) Find the magnitude of each vector

   a) $\mathbf{b} = -7\mathbf{i} + 24\mathbf{j}$

   b) $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$

   ii) Find the dot product of the following vectors.

   a) $|\mathbf{a}| = 15$, $|\mathbf{b}| = 8$ and angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is $30°$

   b) $|\mathbf{a}| = 10$, $|\mathbf{b}| = 8$ and angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is $120°$

5. Find the angle between the two vectors

   i) $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} \quad \mathbf{b} = 5\mathbf{i} - 4\mathbf{j}$

   ii) $\mathbf{a} = -4\mathbf{i} + 7\mathbf{j} \quad \mathbf{b} = 3\mathbf{i} - 6\mathbf{j}$

   iii) $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \quad \mathbf{b} = 6\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$

4.4 The Cross Product (Vector Product)

The video in the link below demonstrates how to apply the vector product method to given vectors.

https://youtu.be/pWbOisq1MJU?si=3hDWulveiHo4F_w4

a) $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k} \quad \mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

b) $\mathbf{a} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k} \quad \mathbf{b} = -7\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$

4.5 Finding the Area of a Triangle and Parallelogram by Using Vector Product

The video in the link below demonstrates how to apply the vector product method to calculate the area of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$.

Where, $\mathbf{a} = 6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{i}$

https://youtu.be/R_PocXgdrcw?si=GMYpT2muVY7LhZkG

4.6 Lines and Planes in Three Dimensions

The video in the link below demonstrates how to find the equation of a straight line in parametric form and provides worked solutions for the following problems.

https://youtu.be/MkjazYnvNP8?si=wwnMK1XvI9HcLbg2

i) Find the vector equation, parametric equations, and symmetric equations for the line that is parallel to the vector $3\mathbf{i}-2\mathbf{j}+5\mathbf{k}$ and passes through the point with coordinates $(2, 4, -3)$.

ii) Find the vector equation, parametric equations and symmetric equations for the line that passes through points $A(1, 3, -2)$ and $B(4, 1, 5)$.

4.7 Determining Whether or Not Given Two Lines Intersect

The video in the link below demonstrates how to check whether two given lines intersect or are parallel. It also demonstrates how to **determine** the coordinates of the point of intersection if the lines intersect. In the end, the video also demonstrates how to determine whether two lines are perpendicular to each other for the following lines, L_1 and L_2.

https://youtu.be/r5DwyBFxD7Q?si=wQgKQjkqj3TTOj-Z

$$L_1 : x = 3 + 2t,\ y = 4 - t,\ z = 1 + 3t$$

$$L_2 : x = 1 + 4s,\ y = 3 - 2s,\ z = 4 + 5s$$

4.8 Determining the Equation of a Plane

The video linked below demonstrates how to find the equation of a plane and then applies this method to a few examples, which are listed below.

https://youtu.be/2sZKZHyaQJ8?si=KDtonXcGic_Ccdlt

i) Find the equation of the plane through the point $(2,\ -5,\ 3)$ and perpendicular to vector $\mathbf{3i} + \mathbf{6j} + \mathbf{5k}$.

ii) Find an equation of the plane through the point $(3,\ 7,\ -2)$ and perpendicular to the vector $\mathbf{3i} + \mathbf{7j} - \mathbf{2k}$

5 Complex Numbers

Objectives

This chapter covers the following topics related to complex numbers. After successful completion of this section, you will be able to:

- Define the symbol i.
- Write down a complex number in Cartesian form, polar form, and exponential form.
- Work with complex numbers.
- Find powers and roots of complex numbers.

5.1 Basic Operations With Complex Numbers

The video in the link below demonstrates basic concepts, including the real and complex components of a complex number, how to find the absolute value and argument of a complex number, and the operations involving complex numbers. It also demonstrates how to solve equations in complex numbers. The solutions to the following questions are demonstrated in the video.

https://youtu.be/OQz1ydBcQSA?si=gnIlST4_5NPIvF-r

i) Plot the following complex numbers in an Argand diagram and calculate their absolute values.

a) $3+4i$ b) $-5+12i$ c) $8-15i$

ii) Simplify and write in complex number form

a) $\sqrt{4}$ b) $\sqrt{-4}$ c) $\sqrt{9}$ d) $\sqrt{-9}$

e) $\sqrt{25}$ f) $\sqrt{-25}$ g) $-\sqrt{25}$ h) $-\sqrt{-25}$

i) $\sqrt{-18}$ j) $-\sqrt{-50}$ k) $\sqrt{-80}$ l) $-\sqrt{-72}$

iii) Simplify

a) i^6 b) i^8 c) i^{12} d) i^{15}

e) i^{29} f) i^{62} g) i^{62} h) i^{201}

iv) Simplify and keep your answer as a single complex number.

a) $(5+2i)+(3+7i)$ b) $(7+3i)+(6+5i)$ c) $(4+8i)-(3-5i)$

d) $7(3+3i)-5(2-6i)$ e) $(4+\sqrt{-25})+(3-\sqrt{-81})$ f) $(7-\sqrt{-9})-(-4+\sqrt{-36})$

g) $(8i)(4i)$ h) $(5i)^2$ i) $(3i)(5i)(-7i)$

j) $(5-3i)(4+7i)$ k) $(6-5i)(3+8i)$ l) $(4+5i)^2$

m) $(5-3i)^3$ n) $(3+4i)(3-4i)$ o) $(5-2i)(5+2i)$

p) $[3(7i)]^2$

v) Dividing complex numbers

a) $\dfrac{4+3i}{5-2i}$ b) $\dfrac{8}{6+i}$ c) $\dfrac{5-2i}{i}$

d) $\dfrac{4+3i}{5-2i}$ e) $\dfrac{3+2i}{7i}$ f) $\dfrac{9}{i^3}$

g) $\dfrac{5-2i}{(2+3i)^2}$ h) $\dfrac{8}{6+i}$ i) $\dfrac{5-2i}{-5+12i}$

vi) Solve following equations

a) $x^2-36=0$ b) $x^2+36=0$ c) $3x^2+48=0$

d) $4x^2+100=0$ e) $-9x^2-100=0$ f) $x^2+\dfrac{1}{9}=0$

g) $3x^2+4x+7=0$ h) $2x^2-3x+9=0$ i) $2x(x-3)=-5$

vii) Further complex numbers problems with solving equations

a) $6x+2i=18+12yi$ b) $3x+4i=15+16yi$ c) $(x+4i)(x-4i)=41$

viii) Find the quadratic equations that comprise the method to convert a given pair of complex roots.

a) $x=3i$ and $x=-3i$ b) $x=4i$ c) $x=4+3i$

d) $x=\sqrt{7}+3i$ e) $x=5-2i$ and $x=5+2i$

ix) Find the roots of the following quadratic equations.

a) $x^2+36=0$ b) $x^2-4x+29=0$

5.2 Polar Form of a Complex Number

i) The video in the link demonstrates how to convert a complex number given in Cartesian form (or rectangular form) into the polar form.

https://youtu.be/jun7m3aTCKE?si=uaKCoDnRvKjyJCsw

ii) The video in the link demonstrates converting the complex numbers given in polar form into Cartesian form (or rectangular form).

https://youtu.be/7eET2C7TRu4?si=4Nfy0Abn6DnENQaK

a) $z = 3\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$ b) $z = 4(\cos 80° + i\sin 80°)$

5.3 Converting a Complex Number Into Exponential Form

The video in the link demonstrates how to convert the following complex numbers into exponential form.

https://youtu.be/RSmFw7sTs9Q?si=J1VqOKNfZlnNWKex

i) Express $z = -1 - \sqrt{3}i$ in the form $z = re^{i\theta}$

ii) Express $z = 5e^{\frac{2\pi}{3}i}$ in the form $z = a + bi$

iii) Write $z = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ in the form $z = re^{i\theta}$

5.4 De Moivre's Theorem

The video in the link demonstrates De Moivre's Theorem and how to use it to convert a complex number into exponential form.

https://youtu.be/LljEyQGUMZY?si=jkN9Zc5V4rEmggCH

Following the introduction, the video goes on to apply the theorem to the example shown below.

Example:

Applying De Moivre's Theorem, simplify

$$\left[3\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right]^4$$

5.5 Applying De Moivre's Theorem to Simplify Powers of Complex Numbers

The video in the link demonstrates how to apply De Moivre's Theorem to simplify complex numbers with powers for the following questions.

https://youtu.be/kJVAvelz1E4?si=ry1HIjE8yrQRispF

a) $(1-\sqrt{3}i)^4$ b) $(-1+\sqrt{3}i)^5$ c) $(-1-i)^{-3}$

5.6 Finding the nth Root of a Complex Number by Using De Moivre's Theorem

The video in the link demonstrates how to apply De Moivre's Theorem to finding the nth root of a complex number.

https://youtu.be/JMnkW8Smf1I?si=c9RmYnFEc2HopjRD

Following the introduction, the video goes on to apply De Moivre's Theorem to the example shown below.

Example:

Find the square roots of $4i$.

6 Integral Calculus

Objectives

This chapter covers the following topics related to integral calculus. After successful completion of this section, you will be able to:

- Evaluate indefinite and definite integrals.
- Use the substitution method and integration-by-parts to evaluate integrals.
- Integration of trigonometric and hyperbolic functions.
- Derive reduction formulae and use these to evaluate integrals.
- Integrate using other methods, such as the method of partial fractions and completing the square.
- Evaluate the area under the curve and between the curves.
- Find the volume of a solid of revolution.
- Determine the lengths of plane curves.
- Find the centre of mass.
- Determine the MacLaurin series expansion for some common functions.
- Define power series and use power series to evaluate integrals.

6.1 Indefinite Integrals

i) The video in the link below demonstrates how to apply the definite integral for the following trigonometric functions

https://youtu.be/M0WcSCxYl2Y?si=LSsjoiIf4ZID0l69

a) $\int \cos x \, dx$ b) $\int \sin x \, dx$ c) $\int \sec^2 x \, dx$

d) $\int \frac{3}{7} \sin x \, dx$ e) $\int \frac{4 \sec^2 x}{5} \, dx$ f) $\int \frac{5}{\sec x} \, dx$

g) $\int \frac{2}{7} \sin x \, dx$ h) $\int \frac{3}{8 \csc x} \, dx$ i) $\int \frac{-4 \cos x}{7} \, dx$

j) $\int 3x - \frac{\sec^2 x}{8} \, dx$ k) $\int 1 + \tan^2 x \, dx$

ii) The video in the link below demonstrates how to apply the definite integral for the following exponential functions.

https://youtu.be/S8-nsBjcnF8?si=CPuFk9-awatIRE40

a) $\int \frac{2}{7}e^{x}dx$ b) $\int \frac{3}{5}e^{2x}dx$ c) $\int \frac{4}{3e^{5x}}dx$

d) $\int \frac{2}{3}e^{3x-2}dx$ e) $\int 3e^{7x} - \frac{2}{3}e^{5x-1}dx$ f) $\int 3e^{5-2x} - \frac{3}{2e^{4x}}dx$

6.2 Definite Integration

Note that answering part m) in this video involves integration by the substitution method, and answering parts n) and o) requires the integration by parts method. These methods are explained in detail under in 6.3 and 6.4.

https://youtu.be/rCWOdFQ3cwQ?si=RkbYC0kDlQD_RPsz

a) $\int_0^4 x^3\,dx$ b) $\int_0^7 7\,dx$ c) $\int_1^2 (3x^2 - 5x + 2)\,dx$

d) $\int_{-1}^2 (2x+3)^2\,dx$ e) $\int_{\frac{1}{2}}^1 \frac{1}{x^2}\,dx$ f) $\int_1^e \frac{1}{x}\,dx$

g) $\int_0^9 \sqrt{x}\,dx$ h) $\int_4^{10} \frac{x^3 - 5x^2}{x}\,dx$ i) $\int_0^3 e^{3x}\,dx$

j) $\int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} \cos x\,dx$ k) $\int_0^{\frac{\pi}{7}} \sin(2x)\,dx$ l) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x\,dx$

m) $\int_0^1 x^2(x^3+5)^2\,dx$ n) $\int_1^2 4xe^{x^2}\,dx$ o) $\int_0^1 xe^{x}\,dx$

6.3 Integration techniques

The following section provides worked examples of integration by substitution and integration by parts techniques for both definite and indefinite integrals.

i) Integration by substitution: Indefinite integrals

https://youtu.be/sdYdnpYn-1o?si=RwZK2wWE8FIjbZ72

a) $\int 4x(x^2+5)^3\,dx$ b) $\int 8\cos(4x)\,dx$ c) $\int x^3e^{x^4}\,dx$

d) $\int 8\sqrt{40-2x^2}\,dx$ e) $\int \frac{x^3}{(2+x^4)^2}\,dx$ f) $\int \sin^4 x\cos x\,dx$

g) $\int \sqrt{5x+4}\,dx$ h) $\int x\sqrt{3x+2}\,dx$ i) $\int 2x\sqrt{4x-5}\,dx$

ii) Integration by substitution: Definite integrals

https://youtu.be/tM4RWc9ryx0?si=nRlNrECo7xo93W0s

a) $\int_0^2 2x(x^2+4)^2\,dx$ b) $\int_0^4 4x\sqrt{16-x^2}\,dx$ c) $\int_1^2 \frac{2x}{(1+x^2)^3}\,dx$

iii) Integration by parts: Indefinite integrals

https://youtu.be/sWSLLO3DS1I?si=1CjLXQcatim3mOf8

a) $\int xe^x\,dx$ b) $\int x\sin x\,dx$ c) $\int x^2\ln x\,dx$

d) $\int \ln x\,dx$ e) $\int x^2\sin x\,dx$ f) $\int x\cos x\,dx$

g) $\int x^2e^x\,dx$ h) $\int (\ln x)^2\,dx$ i) $\int \ln x^7\,dx$

j) $\int e^x\sin x\,dx$ k) $\int \frac{(\ln x)^2}{x}\,dx$ l) $\int e^{3x}\cos(4x)\,dx$

iv) Integration by parts: Definite integrals

https://youtu.be/E6VLoKoJWXY?si=GFv9NsOBgCLML0CH

a) $\int x^2\ln x\,dx$ b) $\int x^2e^x\,dx$

6.4 Integration of Trigonometric Functions

i) Integration by using trigonometric identities

The video in the link below demonstrates how to evaluate the following integrals using Pythagorean and double-angle trigonometric identities.

https://youtu.be/cVl-LUgTka0?si=XgjfTvhzWC2uB9MK

a) $\int \cos^2 x \, dx$ **b)** $\int \tan^2 x \, dx$

c) $\int \cos^4 x \, dx$ **d)** $\int \sin(2x)\cos(2x) \, dx$

ii) Integration by using inverse trigonometric functions

https://youtu.be/J6MvP8SYVfo?si=y5UUpprGXcTZqMzDG

a) $\int \frac{2}{\sqrt{16-x^2}} \, dx$ **b)** $\int \frac{dx}{5+16x^2}$

c) $\int \frac{dx}{x\sqrt{9x^2-4}}$

iii) Integration by using inverse hyperbolic functions

https://youtu.be/xrTTUJYVO-w?si=QoqPWF12Iea8mUrO

a) $\int \frac{x^2+1}{x^4-1} \, dx$ **b)** $\int \frac{3}{x\sqrt{1-\frac{1}{9}x^2}} \, dx$

c) $\int \frac{x}{\sqrt{4x^4+4x^2}} \, dx$ **d)** $\int \left(\frac{1}{1-x}+\frac{1}{1+x}\right) dx$

e) $\int \frac{4+3x}{x\sqrt{1+x^2}} \, dx$

6.5 Reduction Formulae

The following link demonstrates how to use the reduction formula method to evaluate the integrals listed below.

https://youtu.be/an8vp2QhONE?si=rf0zIpCvtyt-AEVi

a) $\int \cos^n x \, dx$ **b)** $\int_0^{\frac{\pi}{2}} x^n \sin x \, dx$

c) $\int \frac{x^n}{\sqrt{x+1}} \, dx$ **d)** $\int x(\ln x)^{2n} \, dx$

6.6 Integration by using partial fractions

i) The link below demonstrates how to find the partial fractions for the following improper rational functions, but **not the integral**.

https://youtu.be/MKe71wzJIGY?si=369wMYg4jkPYcwkj

a) $\dfrac{x^2+8x+13}{(x+3)(x+2)}$

b) $\dfrac{2x^3+3x^2-2}{(x+2)(x-1)}$

c) $\dfrac{2x^2+17x+11}{(2x+1)(x+2)}$

d) $\dfrac{x^3-x^2-16x-13}{x^2-3x-10}$

ii) The link below demonstrates how to evaluate integrals of the following rational functions by using partial fractions.

https://youtu.be/6rXByMcuAyI?si=13oLesnUMTnaS4Ya

a) $\displaystyle\int \frac{1}{x^2-4}\,dx$

b) $\displaystyle\int \frac{x-4}{x^2+2x-15}\,dx$

c) $\displaystyle\int \frac{x}{(x-1)(x-2)^2}\,dx$

d) $\displaystyle\int \frac{x^2+9}{(x^2-1)(x^2+4)}\,dx$

iii) Integrating by completing the square

The video in the link below demonstrates how to evaluate integrals using the completing the square method for the questions listed below.

https://youtu.be/vQerALNpNfo?si=PHAV2Bf6NFWrb7sq

a) $\displaystyle\int \frac{dx}{x^2-6x+13}$

b) $\displaystyle\int \frac{x-5}{x^2+8x+22}\,dx$

6.7 Evaluate the Area Under the Curve and Between the Curves

The video in the link below demonstrates how to find the area between a curve and the X-axis, between two curves along the X-axis, between a curve and the Y-axis and between two curves along the Y-axis.

https://youtu.be/kgg5Rspf1Js?si=goJv-vaaB1z9TqzQ

Then, it demonstrates how to evaluate the area of the region bounded by:

a) the line with equation $y = 8 - 2x$, the X-axis and the Y-axis.

b) the line with equation $y = x$ and the curve with equation $y = x^2$

c) the curves with equations $y = x^2$ and $x = y^2$

d) the curves with equations $x = 1 - y^2$ and $x = y^2 - 1$

e) the curve with equation $y = x^2 - 4x$ and the X-axis

f) the line with equation $y = 6 - 3x$ and the curve with equation $y = x^2 - 4x$

g) the line with equation $x = 3y - 2$ and the curve with equation $x = 2y^2 - 4$

6.8 Volume of a Solid of Revolution

The video in the link below demonstrates the methodology of finding the volume of revolution along the X-axis and Y-axis.

It also demonstrates evaluating the volume of the solid generated by the following curves between given values:

https://youtu.be/SAHSVg7Jw_A?si=QZvRPH864CB6-OPH

a) $y = \sqrt{x}$ rotating along the X-axis for 360° between $x = 0$ and $x = 4$

b) $y = \dfrac{1}{x}$ rotating along the X-axis for 360° between $x = 1$ and $x = 3$

c) $y = x^2$ rotating along the Y-axis for 360° between $y = 0$ and $y = 4$

d) $y = \dfrac{x^3}{2}$ rotating along the Y-axis for 360° between $x = 0$ and $y = 1$

6.9 The Length of a Curve

The video in the link below demonstrates how to evaluate the length of a curve for the following examples.

https://youtu.be/DNDAwWIL5FY?si=UzLCyBPEw_M-g8OL

a) $f(x) = 1 + 6x^{\frac{3}{2}},\ 0 \leq x \leq 1$

b) $f(x) = \frac{3}{2}x^{\frac{2}{3}},\ 1 \leq x \leq 8$

c) $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}},\ 0 \leq y \leq 4$

5.10 Centre of Mass

The video that is comprised in the link below demonstrates the concept of the moment of a point mass about a point and extends the definition to a system of point masses about

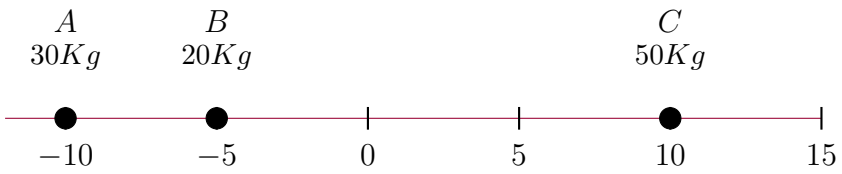
the origin. The centre of mass is then defined, and the physical significance is explained in terms of doors and see-saws.

https://youtu.be/ZxOIlCjh_pw?si=v9f-Mmzawm9NoOLR

i) Moreover, the video also explains how to find the answer to the following question.

Two particles with masses 30 kg, 20 kg, and 50 kg are attached at points A, B, and C on a light, uniform rod.

The rod is supported at point $O(0,\ 0)$, acting as a pivot.



(a) Will the system be balanced about point O?

(b) If not, where should the fulcrum (pivot) be placed to achieve balance?

ii) The video in the link below demonstrates the method of finding the centre of mass for a non-uniform rod. It also demonstrates how to find the centre of mass of two examples that are listed below.

https://youtu.be/xxXUCYzdrlM?si=j2IShhK74DCjiNmB

(a) Find the centre of mass of a uniform rod of length L with constant density ρ.

(b) Find the centre of mass of a non-uniform rod of length L with density x.

iii) The video below demonstrates how to prove that the centre of mass of a uniform rod with mass M is at its middle.

https://youtu.be/HCIhzGG_ZVY?si=we3XZxB4eT34zzUa

7 Matrix Algebra

Objectives

This chapter covers the following topics related to matrices. After successful completion of this section, you will be able to:

- Evaluate 2 x 2 and 3 x 3 determinants.
- Define what is meant by a "cofactor".
- Use determinants to solve simultaneous equations.
- Operate with matrices.
- Find the inverse of a matrix.
- Use matrices to solve simultaneous equations.
- Use Gaussian elimination to solve a set of linear equations.
- Use row operations to find the inverse of a matrix.
- Find the eigenvalues and corresponding eigenvectors for a matrix.

The video in the link below illustrates the properties of matrices and their fundamental operations.

https://youtu.be/cMsubZQEg3A?si=Hn5NIKo2-OpAeO93

7.1 Introduction to Matrices and Basic Operations

The following videos demonstrate the **cofactor** method and explain the step-by-step process for finding the solution to specific questions.

i) The video in the link below demonstrates the methodology for finding the **determinant** of 2x2 and 3x3 matrices and shows how to find the determinants for the following particular matrices.

https://youtu.be/3ROzG6n4yMc?si=_IMTlzoVWruGr3w3

2×2 matrices: **a)** $\begin{pmatrix} 3 & 5 \\ -4 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} -7 & 8 \\ 4 & -3 \end{pmatrix}$

3×3 matrices: **a)** $\begin{pmatrix} 2 & 4 & -3 \\ 5 & 7 & 6 \\ -8 & 1 & 9 \end{pmatrix}$ **b)** $\begin{pmatrix} 5 & 7 & -8 \\ 4 & -3 & 6 \\ 1 & 7 & -9 \end{pmatrix}$

ii) The video in the link below demonstrates how to find the **determinant** of the following 4x4 matrix.

https://youtu.be/fWzUwrt1Z0s?si=N8DsuJsutPU9u4bU

a) $\begin{pmatrix} 1 & 0 & 4 & -6 \\ 2 & 5 & 0 & 3 \\ -1 & 2 & 3 & 5 \\ 2 & 1 & -2 & 3 \end{pmatrix}$

7.2 Cramer's Rule

The video in the link below demonstrates how to solve the following simultaneous equations with two and three variables using the Cramer rule.

https://youtu.be/RdLo-9jh2EM?si=VLpejFdANIL4uIqy

i) $\begin{aligned} X_1 + 3X_2 &= 5 \\ 2X_1 + 2X_2 &= 6 \end{aligned}$

ii) $\begin{aligned} 2X_1 + X_2 + X_3 &= 1 \\ 3X_2 - X_3 &= 7 \\ X_1 + 2X_2 + 2X_3 &= -4 \end{aligned}$

iii) $\begin{aligned} 2x + y + z &= 3 \\ x - y - z &= 0 \\ x + 2y + z &= 0 \end{aligned}$

7.3 Transpose of a Matrix

The video in the link below demonstrates how to find the transpose matrix for the matrices listed below.

https://youtu.be/pXp3nQ5exms?si=68GUbFlB5ry5W3Bg

a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

7.4 The Inverse of a Square Matrix Using the Adjoint Matrix Method

The video of the link below demonstrates how to find the inverse of the following 2x2 and 3x3 square matrices.

https://youtu.be/SRVrA-j6J1U?si=E7L7Q6U0QqiOZSol

a) $\begin{pmatrix} 2 & -4 \\ 3 & 5 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

7.5 Solving Systems of Linear Equations

The video in the link below demonstrates, with examples, what is meant by **one solution, many solutions, no solution, consistent, inconsistent,** and **independent and dependent cases of systems of linear equations**.

https://youtu.be/bq5gDsEdN3Q?si=pKxtOkHDOZuGM612

7.6 Gaussian Reduction

The video links below demonstrate how to solve systems of linear equations using the Gaussian elimination method.

(a) **Unique Solution:** The link below demonstrates how to solve systems of linear equations with unique solutions.

https://youtu.be/eDb6iuqi6Uk?si=sx03zPtnxyDllav_

$$\begin{aligned} x+y-z &= -2 \\ 2x-y+z &= 5 \\ x+2y+2z &= 1 \end{aligned} \qquad \begin{aligned} 2x+y-z &= 1 \\ 3x+2y+z &= 10 \\ 2x-y+2z &= 6 \end{aligned}$$

(b) **Infinite solutions:** The link below demonstrates how to solve the following systems of linear equations with infinitely many solutions.

https://youtu.be/0H9FLpfsjrI?si=0xOtMbrzgunz0jAi

$$\begin{aligned} x-3y+2z &= 1 \\ 4x+y-5z &= 17 \\ 2x-3y+z &= 5 \end{aligned}$$

(c) **No solution:** The link below demonstrates how to solve the following systems of linear equations with no solutions.

https://youtu.be/aRXHoRHnCgg?si=qVmS5KjVth01Szh4

$$\begin{aligned} 3x+y+z &= 0 \\ x+2y-3z &= 0 \\ 3x-4y+11z &= 0 \end{aligned}$$

7.7 Homogeneous Linear Systems

The following video demonstrates homogeneous systems of linear equations, including both trivial and non-trivial solutions, with examples below.

https://youtu.be/JlJWyWJARRU?si=u3V8CgxAp4yGyOGQ

$$\begin{aligned} 5x + 4y &= 0 \\ 2x - 2y &= 0 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 6x_2 - 3x_3 &= 0 \\ x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

7.8 Inversion of Matrices by Reduction

The following video demonstrates how to find the inverse of matrix A, as given below, using the row reduction method.

https://youtu.be/Fg7_mv3izR0?si=qMxabu98V3Ys8O4m

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

7.9 Eigenvalues and Eigenvectors

The following video demonstrates how to find the eigenvalues and eigenvectors of the 3x3 matrix for matrix A.

https://youtu.be/9kfp85NpqfE?si=B0u-HUjPPSJGB3KO

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{pmatrix}$$

8 Probability and Statistics

Objectives

This chapter covers the following topics related to probability and statistics. After successful completion of this section, you will be able to:

- Understand and apply statistical terminology.
- Evaluate the mean, median, mode, range, standard deviation, and variance for a given set of data.
- Distinguish between discrete and continuous random variables.
- Determine if a probability distribution is valid or not.
- Recognise when a random variable follows a binomial distribution and use this distribution to calculate probabilities.
- Recognise when a random variable follows a Poisson distribution and use this distribution to calculate probabilities.
- Calculate probabilities using the standard normal distribution.

8.1 Understand and Apply Statistical Terminology

The video in the link below explains the basic statistical terms that students should be familiar with. It also provides some real-life examples to help you understand the meaning of each term.

https://youtu.be/kxH9WlHEers?si=zqQYFgJMN14eUtuk

8.2 Evaluate the Mean, Median, Mode, Range, Standard Deviation, and Variance for a Given Set of Data

i) The video in the link below demonstrates how to find the median, mode, range, mean, variance and the standard deviation.

https://youtu.be/179ce7ZzFA8?si=LzkR2BK3YI3YCSZ9

ii) The video in the link below demonstrates what is meant by the standard deviation and how imperative the standard deviation is with regard to the data set that you have. It also explains the distinguish between sample standard deviation and population standard deviation.

https://youtu.be/esskJJF8pCc?si=JJRxf4dGFGp-Vb3X

8.3 Discrete and Continuous Random Variables

The video in the link demonstrates the distinction between discrete and continuous random variables. It also provides specific examples for determining whether data is discrete or continuous from a list of given variables.

https://youtu.be/cz4nPSA9rlc?si=cDh9evzMTniNwgd6

8.4 Probability Distributions

1. **Discrete probability distributions:** The video in the link below demonstrates how to construct a discrete probability distribution and verify its validity.

   https://youtu.be/UnzbuqgU2LE?si=s05dKEQwabPwbTre

   The video in the link below demonstrates how to find the following discrete random variables problems.

   https://youtu.be/xU3ppOEWY3c?si=CnAnLxilzY8XkeFm

   i) A discrete random variable X has the following probability distribution:

x	1	2	3	4	5
$P(X = x)$	0.2	0.1	0.1	k	k

   (a) Find the value of k

   (b) Find $P(X > 2)$

   (c) Find $P(2 < X \leq 4)$

   ii) A discrete random variable X has the following probability distribution:

$$P(X = x) = \begin{cases} 0.1 & \text{if } x = 1, 2 \\ 0.2 & \text{if } x = 3, 4 \\ k & \text{if } x = 5 \end{cases}$$

   (a) Find the value of k

   (b) Draw a table giving the probability distribution of X

   iii) A discrete random variable X has the following probability distribution:

$$P(X = x) = \begin{cases} kx & \text{for } x = 1,\ 2,\ 3 \\ k(x-1) & \text{for } x = 4,\ 6 \end{cases}$$

(a) Find the value of k

(b) Draw a table giving the probability distribution of X

iv) A discrete random variable X has the following probability distribution:

x	1	2	3	4	5
$P(X = x)$	0.15	0.25	0.1	0.3	0.2

(a) Draw a table showing the cumulative function $F(X)$

(b) Write down $F(2)$

(c) Write down $F(3.1)$

2. Continuous probability distributions: The video in the link demonstrates how to construct a continuous probability distribution and verify its validity.

https://youtu.be/9KVR1hJ8SxI?si=qJWLr-uANajnRczy

i) Find the value of k for which the following function would be a valid probability density function.

$$f(x) = \begin{cases} k(x^2+1) & \text{if } 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

ii) A probability density function of a random variable X is given below.

$$f(x) = \begin{cases} \frac{1}{24}(x^2+1) & \text{if } 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Calculate $P(X = 3)$.

b) Calculate $P(2 < X \leq 3)$.

iii) A probability dencity function of a random variable X is given below.

Find $P(2 \leq X \leq 4)$.

8.5 Binomial Distribution:

The Binomial Distribution is a discrete probability distribution. The video in the link demonstrates how to determine whether a given discrete distribution is binomial and outlines its key characteristics.

https://youtu.be/NaDZ0zVTyXQ?si=Q6oQH0BqH0DYGDdx

The video in the link demonstrates how to calculate probabilities using the Binomial Distribution for the examples listed below.

https://youtu.be/3PWKQiLK41M?si=q360lIgbWM0728kl

i) A six-sided die is rolled 12 times. What is the probability of getting a 4 five times?

ii) A multiple choice test contains 20 questions with answer choices A, B, C, and D. Only one answer choice to each question represents a correct answer. Find the probability that a student will answer exactly 6 questions correct if he makes random guesses on all 20 questions.

iii) 25% of all students enrolled in high school XYZ are taking Algebra. 30 students are chosen at random.

- **a)** Find the probability that exactly 7 students out of the 30 chosen are taking Algebra.
- **b)** What is the probability that fewer than 5 students out of the 30 who are selected are taking Algebra?
- **c)** Calculate the mean and standard deviation of this binomial distribution.

8.6 Poisson Distribution

The Poisson Distribution is a discrete probability distribution. The video in the link demonstrates how to determine whether a given discrete distribution is Poisson and outlines its key characteristics.

https://youtu.be/jmqZG6roVqU?si=S8iN2PL1bjCbS6No

The video also demonstrates how to use the Poisson distribution to solve the example provided below.

Example:

One nanogram of Plutonium-239 will have an average of 2.3 radioactive decays per second, and the number of decays will follow a Poisson distribution.

What is the probability that in a 2 second period there are exactly 3 radioactive decays?

8.7 Normal Distribution

The video in the link introduces the characteristics of the Normal Distribution and demonstrates how to find the answer to the example outlined below.

https://youtu.be/xI9ZHGOSaCg?si=zvZ9v9SovR4NDKUa

Example:

Your local pizza shop claims that the largest pizza they sell is at least 16 inches, or it's free. Given that the size of their pizza is normally distributed with $\mu = 16.3$ inches and $\sigma = 0.2$ inches.

a) What's the probability of getting free pizza?

b) What's the probability of getting lucky with a pizza over 16.5 inches?

c) What's the probability of getting pizza between 15.95 and 16.63 inches?